

Research Article

Hassan Akhouayri, Maïtine Bergounioux*, Anabela Da Silva, Peter Elbau, Amelie Litman and Leonidas Mindrinos

Quantitative thermoacoustic tomography with microwaves sources

DOI: 10.1515/jiip-2016-0012

Received February 5, 2016; revised October 10, 2016; accepted November 3, 2016

Abstract: We investigate a quantitative thermoacoustic tomography process. We aim to recover the electric susceptibility and the conductivity of a medium when the sources are in the microwaves range. We focus on the case where the source signal has a slow time-varying envelope. We present the direct problem coupling equations for the electric field, the temperature variation and the pressure (to be measured via sensors). Then we give a variational formulation of the inverse problem which takes into account the entire electromagnetic, thermal and acoustic coupled system, and perform the formal computation of the optimality system.

Keywords: Thermoacoustic tomography, inverse problem, optimal control, Maxwell's equations

MSC 2010: 35M33, 35Q61, 49N45, 80A23, 93C20

1 Introduction

In this paper we investigate a model to describe the so-called thermoacoustic tomography process as precisely as possible, in the very case where the sources are in the microwaves range [13, 20, 27, 35]. Let us briefly recall the principle of photo/thermoacoustic tomography (we refer to [21, 22, 28, 33, 34] for a more complete description).

Photoacoustic and thermoacoustic tomographies (PAT and TAT) consist in transmitting a time-modulated electromagnetic wave (generally a pulse in the optical or near-infrared domain for PAT and in the microwave domain for TAT), whose energy is locally absorbed by the tissue absorbers, transformed into heat and then into acoustic (generally ultrasonic) waves via the thermo-expansion mechanism.

The physical model routinely used in PAT is similar to the one in TAT, apart from the fact that the reconstructions performed are based on input signals in different frequency bands. In TAT, one reconstructs refraction indices and absorption coefficients, in PAT, one reconstructs diffusion and absorption coefficients.

Most of quantitative PAT methods assume that the initial pressure distribution has been reconstructed and then perform the identification of the absorption and/or diffusion coefficients. To achieve this goal one can use many measurements, for instance different lightning sources [4] or different wavelength sources [37]. Some authors assume that the Green function associated to the acoustic wave is known and perform the inversion with the pressure measurements [15, 31, 32]. Most models are based on the diffusion equation but there are some works that use the radiative transfer equation (RTE) as in [30]. The simplified model from PAT

Hassan Akhouayri, Anabela Da Silva, Amelie Litman: Aix-Marseille Université, CNRS, Centrale Marseille, Institut Fresnel UMR7249, 13013 Marseille, France, e-mail: hassan.akhouayri@fresnel.fr, anabela.dasilva@fresnel.fr, amelie.litman@fresnel.fr

***Corresponding author: Maïtine Bergounioux:** University of d'Orléans, Laboratory MAPMO, CNRS, UMR 7349, Fédération Denis Poisson, FR 2964, Bâtiment de Mathématiques, BP 6759, 45067 Orléans cedex 2, France, e-mail: maitine.bergounioux@univ-orleans.fr

Peter Elbau, Leonidas Mindrinos: Computational Science Center, University of Vienna, 1090 Vienna, Austria, e-mail: peter.elbau@univie.ac.at, leonidas.mindrinos@univie.ac.at

that only involves the pressure effects, does not take into account the optical inhomogeneity of the medium. It is then necessary to go one step further in the modelling of the medium, which will allow to reconstruct absorption coefficients. This has been achieved in [7] where the full coupled model has been investigated.

The aim of this paper is to focus on the TAT case and to provide a complete model coupling electromagnetic source effects, heat transfer and pressure equations, for quantitative thermoacoustic tomography. Though the equations are well known, such a model that fully describes the whole direct problem is new to our knowledge. The heat transfer equations were first presented in the context of PAT in [10] but then neglected under the assumption of instantaneous heating. Indeed, the quantitative PAT methods, currently available, assume that the initial pressure distribution, i.e., the amount of absorbed radiation, has been first reconstructed and then perform the identification of the absorption and diffusion coefficients using mainly the diffusion approximation of the radiative transfer equation, see for example [1, 3, 16, 26, 38], or the electromagnetic parameters using Maxwell's equations [12].

For quantitative reconstructions in TAT, we refer to [5] where the authors neglect the thermal effects and handle the scalar Helmholtz equation. Furthermore, they use a linearized version to deal with the inverse problem to recover the absorption coefficient. In [6], they present a similar approach for the vectorial Maxwell's equation and reconstruct the absorption coefficient and the refractive index of the materials. In [2], the Helmholtz equation is considered and an optimal control algorithm is proposed to recover the absorption coefficient from the local knowledge of the absorbed radiation. An iterative scheme, using the Helmholtz equation, to reconstruct the conductivity is also presented in [17].

In this paper, we present the direct problem, where the thermal effects are fully kept. Nevertheless, we focus on the very case where the source pulse has a slowly time varying envelope, which is the most relevant situation from a practical point of view [24, 36]. We also propose a way to solve the corresponding quantitative inverse problem using a variational formulation as in [7], thus taking into account the complete coupled model. The paper is organized as follows: we first set the description of the different processes, namely, the electromagnetic, heating and acoustic ones. Then we specify the mathematical model and present the equivalent minimization problem. The paper ends with the (formal) optimality conditions for the discretized system of equations.

2 The thermoacoustic model

2.1 Electromagnetic propagation

To describe the propagation of the electromagnetic waves due to an illumination by a radio-frequency source, we consider the classical macroscopic Maxwell's equations (in SI units) and the constitutive relations between the electric displacement D , the electric field E , the magnetic induction B and the magnetic field H . All these vectors are three-dimensional real-valued functions of position x and time t .

2.1.1 Maxwell's equations

We assume that the medium is *non-magnetic, linear dielectric, isotropic and inhomogeneous*. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary such that the exterior domain $\mathbb{R}^3 \setminus \Omega$ is connected. The domain Ω contains the support of the inhomogeneity we aim at recovering with the TAT technique. This domain may be possibly disconnected.

In the sequel,

- the external current density J^{ex} satisfies $\text{supp } J^{\text{ex}}(t, \cdot) \cap \Omega = \emptyset$ for any $t > 0$ and is assumed to be known.
- $\chi : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$, $(t, x) \mapsto \chi(t, x)$, consists in a combination of the electric susceptibility ξ and the conductivity σ which may vary according to the medium (this will be specified in the sequel). The *exterior* part $\chi^{\text{ex}} : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is assumed to be constant (but not necessarily equal to 0) with respect to space in

the exterior domain, the so-called *background medium*. Then we can identify Ω as the support of $\chi - \chi^{\text{ex}}$. This implies that $\hat{\chi} - \hat{\chi}^{\text{ex}}$ has (spatial) compact support as well. Here $\hat{\chi}$ denotes the Fourier transform of χ with respect to time (whose definition is recalled in the sequel in equation (2.1)). In addition, we assume $\chi(t, \cdot) = 0$ and $\chi^{\text{ex}}(t, \cdot) = 0$ for $t < 0$.

- ϵ_0, μ_0 are the known permittivity and permeability of vacuum, respectively. One can see examples of typical values of all the physical quantities in Section 2.3.
- $c_0 = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light in vacuum.
- as the medium is non-magnetic, we set the magnetic permeability μ of the medium to $\mu(t, x) = \mu_0$ for every $(t, x) \in \mathbb{R} \times \mathbb{R}^3$.

Let us now recall Maxwell's equations. For every $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, the electric field E , the magnetic field H and the electric displacement D satisfy

$$\begin{aligned}\operatorname{curl}_x E(t, x) &= -\mu_0 \frac{\partial H}{\partial t}(t, x), \\ \operatorname{curl}_x H(t, x) &= \frac{\partial D}{\partial t}(t, x) + J(t, x),\end{aligned}$$

where we decompose the electric current density J into $J := J^{\text{in}} + J^{\text{ex}}$. The interior part J^{in} corresponds to the current density generated by ohmic losses while J^{ex} is due to the electromagnetic primary excitation.

As the medium is considered to be linear, isotropic, inhomogeneous and non-magnetic, we get, for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$D(t, x) = \epsilon_0 \left(E(t, x) + \int_0^\infty \xi(\tau, x) E(t - \tau, x) d\tau \right) = \epsilon_0 (E + \xi *_t E)(t, x),$$

where $*_t$ denotes the time convolution operator (indeed $\xi(t, \cdot) = 0$ for $t < 0$). This reads, in the frequency domain,

$$\hat{D}(\omega, x) = \epsilon_0 (1 + \hat{\xi}(\omega, x)) \hat{E}(\omega, x), \quad (\omega, x) \in \mathbb{R} \times \mathbb{R}^3,$$

where \hat{f} stands for the Fourier transform of a function f with respect to t

$$\hat{f}(\omega, x) = \int_{-\infty}^{\infty} f(t, x) e^{i\omega t} dt, \quad (2.1)$$

and $i^2 = -1$. Similarly,

$$J^{\text{in}}(t, x) = \int_0^\infty \sigma(\tau, x) E(t - \tau, x) d\tau = (\sigma *_t E)(t, x), \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3,$$

or in the frequency domain,

$$\hat{J}^{\text{in}}(\omega, x) = \hat{\sigma}(\omega, x) \hat{E}(\omega, x), \quad (\omega, x) \in \mathbb{R} \times \mathbb{R}^3.$$

Thus, the exact definition of χ is given very easily in the frequency domain by

$$\hat{\chi}(\omega, x) = \hat{\xi}(\omega, x) + \frac{i}{\epsilon_0 \omega} \hat{\sigma}(\omega, x), \quad (\omega, x) \in \mathbb{R} \setminus \{0\} \times \mathbb{R}^3. \quad (2.2)$$

Remark 2.1. We do not specify the mathematical framework by now. We assume that χ belongs to a suitable space that allows to define $\hat{\chi}$. This is the case, for example, if $(\xi(\cdot, x), \sigma(\cdot, x)) \in L^2(\mathbb{R})$ and σ is differentiable (with respect to time).

For any given ω , the electromagnetic fields satisfy for any point x in \mathbb{R}^3

$$\begin{aligned}\operatorname{curl}_x \hat{E}(\omega, x) &= i\omega\mu_0 \hat{H}(\omega, x), \\ \operatorname{curl}_x \hat{H}(\omega, x) &= -i\omega\epsilon_0 (1 + \hat{\chi}(\omega, x)) \hat{E}(\omega, x) + \hat{J}^{\text{ex}}(\omega, x),\end{aligned}$$

that is, for every $(\omega, x) \in \mathbb{R} \times \mathbb{R}^3$,

$$\operatorname{curl}_x \operatorname{curl}_x \hat{E}(\omega, x) - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}(\omega, x)) \hat{E}(\omega, x) = i\omega\mu_0 \hat{J}^{\text{ex}}(\omega, x). \quad (2.3)$$

If we assume that the support of all the sources J^{ex} is bounded and that the sources and the inhomogeneities are all located within a finite distance from the origin of the coordinate system, then boundary conditions at infinity are provided by the Silver–Müller radiation condition

$$\lim_{|x| \rightarrow \infty} \left(x \times \text{curl}_x \hat{E} - i \frac{\omega}{c_0} \sqrt{1 + \hat{\chi}^{\text{ex}} |x|} \hat{E} \right) = 0, \quad (2.4)$$

uniformly in $x/|x|$. Here, $a \times b$ denotes the \mathbb{R}^3 vectorial product, while $a \cdot b$ denotes the \mathbb{R}^3 scalar product and $|a|$ the (euclidean) associated norm of vectors $a, b \in \mathbb{R}^3$.

2.1.2 Heating function

Let us now focus on the heating function (density of absorbed power (W/m^3)) caused by the electromagnetic source excitation. The total electromagnetic energy density is given by $\mathbf{E} = \frac{1}{2}(E \cdot D + \mu_0 H \cdot H)$. By virtue of Maxwell's equations, we get for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$-\text{div}_x Q(t, x) = \mu_0 \left(H \cdot \frac{\partial H}{\partial t} \right)(t, x) + \left(E \cdot \frac{\partial D}{\partial t} \right)(t, x) + (E \cdot J)(t, x),$$

where $Q(t, x) = E(t, x) \times H(t, x)$ is the Poynting vector [18]. Therefore, considering Maxwell's equations, we obtain

$$-\text{div}_x Q(t, x) = \frac{1}{2} \left(E \cdot \frac{\partial D}{\partial t} - \frac{\partial E}{\partial t} \cdot D \right)(t, x) + \frac{\partial \mathbf{E}}{\partial t}(t, x) + (E \cdot J)(t, x).$$

The absorbed power out of the electromagnetic source excitation is given by

$$\Pi(t, x) := \frac{1}{2} \left(E \cdot \frac{\partial D}{\partial t} - \frac{\partial E}{\partial t} \cdot D \right)(t, x) + (E \cdot J^{\text{in}})(t, x),$$

since the term $\frac{\partial \mathbf{E}}{\partial t}$ describes the rate of change of the energy and the term J^{ex} only concerns the external sources.

However, due to the difference between the electromagnetic and thermal time scales, the absorbed density Π_a has to be computed by the mean of an average on a time interval of critical length T_c sufficiently large,

$$\Pi_a(t, x) := \langle \Pi \rangle_{T_c}(t, x) = \frac{1}{T_c} \int_{t-T_c}^t \Pi(\tau, x) d\tau. \quad (2.5)$$

2.1.3 Slowly time-varying envelope approximation

The critical time T_c depends on the time scale coupling the different phenomena (electromagnetic, thermal and acoustic). Indeed, the acoustic and the electromagnetic waves have different wave velocities. From a numerical point of view, T_c will give the ratio between the time steps of acoustic and electromagnetic computations. This average value can be considered to be equal to the rate of the energy density deposited in the material at position x , during the total duration of one *period* or *pseudo-period* of the illumination process. It can be made precise in the case where the electromagnetic signal presents a slowly time-varying envelope. This assumption makes sense when the high-frequency electromagnetic excitation signal is modulated by a pulse whose temporal width is several decades of times longer than the high-frequency oscillation period $T_c = 2\pi/\omega_c$. In that case, the source has a narrow bandwidth in $W_c := [\omega_c - \Delta_c, \omega_c + \Delta_c]$. Such situations occur in microwave setups, which rely on high power pulse-modulated carrier frequency generators, usually operating in the GHz frequency range, with pulse duration in the μs range in order to ensure sufficient energy deposition into the sample, with the drawback of reducing the spatial resolution to the mm range [9, 19, 20, 24, 27, 29, 35, 36].

The excitation density current takes the form

$$J^{\text{ex}}(t, x) = \Re(j^{\text{ex}}(t, x)e^{-i\omega_c t}), \quad (2.6)$$

where \Re stands for the real part and the complex amplitude j^{ex} has a slowly varying envelope on a short period T_c . Let us recall that if $A(t, x) = \Re(a(t, x)e^{-i\omega_c t})$ is a signal with a slowly varying envelope, then

$$\left| \frac{\partial^2 a}{\partial t^2}(t, x) \right| \ll \omega_c \left| \frac{\partial a}{\partial t}(t, x) \right| \ll \omega_c^2 |a(t, x)|, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3.$$

Note that the above relation comes from

$$\frac{\partial^2(a(t, x)e^{-i\omega_c t})}{\partial t^2} = \left[\frac{\partial^2 a(t, x)}{\partial t^2} - 2i\omega_c \frac{\partial a(t, x)}{\partial t} - \omega_c^2 a(t, x) \right] e^{-i\omega_c t}.$$

In practice, the rate of the energy density time-averaged over the short period T_c is of interest. Before deriving the form of Π_a for this specific form of J^{ex} , we address the following properties:

- The electromagnetic properties of the medium, given by χ or equivalently by ξ and σ , are assumed to be constant with respect to the frequency in the range W_c and depend only on the central frequency, namely,

$$\tilde{\xi}(\omega, x) \simeq \tilde{\xi}(\omega_c, x), \quad \hat{\sigma}(\omega, x) \simeq \hat{\sigma}(\omega_c, x), \quad \omega \in W_c.$$

- If the current density is given by equation (2.6), then the quantities E , D and J^{in} admit the same decomposition with slowly varying amplitude:

$$\begin{aligned} E(t, x) &= \Re(e(t, x)e^{-i\omega_c t}), & e(t, x) &= \frac{e^{i\omega_c t}}{\pi} \int_0^\infty \hat{E}(\omega, x)e^{-i\omega t} d\omega, \\ D(t, x) &= \Re(d(t, x)e^{-i\omega_c t}), & d(t, x) &= \frac{e^{i\omega_c t}}{\pi} \int_0^\infty \hat{D}(\omega, x)e^{-i\omega t} d\omega, \\ j^{\text{in}}(t, x) &= \Re(j^{\text{in}}(t, x)e^{-i\omega_c t}), & j^{\text{in}}(t, x) &= \frac{e^{i\omega_c t}}{\pi} \int_0^\infty \hat{j}^{\text{in}}(\omega, x)e^{-i\omega t} d\omega. \end{aligned}$$

This follows from the unique solvability of Maxwell's equations and the fact that \hat{E} , \hat{D} and \hat{j}^{in} are also supported in W_c . In particular,

$$d(t, x) \simeq \epsilon_0(1 + \tilde{\xi}(\omega_c, x))e(t, x).$$

This comes from

$$\begin{aligned} d(t, x)e^{-i\omega_c t} &= \frac{1}{\pi} \int_0^\infty \hat{D}(\omega, x)e^{-i\omega t} d\omega \\ &= \frac{1}{\pi} \int_{W_c} \epsilon_0(1 + \tilde{\xi}(\omega, x))\hat{E}(\omega, x)e^{-i\omega t} d\omega \\ &\simeq \frac{\epsilon_0}{\pi} (1 + \tilde{\xi}(\omega_c, x)) \int_{W_c} \hat{E}(\omega, x)e^{-i\omega t} d\omega \\ &= \epsilon_0(1 + \tilde{\xi}(\omega_c, x))e(t, x)e^{-i\omega_c t}. \end{aligned}$$

Similarly,

$$j^{\text{in}}(t, x) \simeq \hat{\sigma}(\omega_c, x)e(t, x).$$

- Let $A(t, x) = \Re(a(t, x)e^{-i\omega_c t})$ and $B(t, x) = \Re(b(t, x)e^{-i\omega_c t})$ be two slowly varying envelope signals. We obtain for the averaged value over T_c for the product of these two functions

$$\langle A \cdot B \rangle_{T_c}(t, x) \simeq \frac{1}{2} \Re(a(t, x) \cdot \overline{b(t, x)}), \quad (2.7)$$

since

$$\begin{aligned} A \cdot B &= \frac{1}{4} (ae^{-i\omega_c t} + \overline{ae^{-i\omega_c t}}) \cdot (be^{-i\omega_c t} + \overline{be^{-i\omega_c t}}) \\ &= \frac{1}{4} (a \cdot be^{-i2\omega_c t} + \overline{a \cdot be^{-i2\omega_c t}}) + \frac{1}{4} (a \cdot \bar{b} + \bar{a} \cdot b). \end{aligned}$$

Here \bar{a} stands for the complex conjugate of a . The first two terms correspond to a signal with high frequency components while the last two terms are slowly varying with respect to t . Thus, their short-time average value is given by

$$\langle A \cdot B \rangle_{T_c}(t, x) \simeq \frac{1}{4} \langle a \cdot \bar{b} + \bar{a} \cdot b \rangle_{T_c}(t, x),$$

resulting in (2.7). One can also show that $\langle A \cdot \frac{\partial A}{\partial t} \rangle_{T_c}(t, x) \simeq 0$.

Therefore, the short-time averaged absorbed energy Π_a defined in equation (2.5), using (2.2), (2.7) and the above assumptions, is given by

$$\Pi_a(t, x) \simeq \frac{\epsilon_0 \omega_c}{2} \Im(\hat{\chi}(\omega_c, x)) |e(t, x)|^2 = \frac{\epsilon_0 \omega_c}{2\pi^2} \Im(\hat{\chi}(\omega_c, x)) \left| \int_0^\infty \hat{E}(\omega, x) e^{-i\omega t} d\omega \right|^2. \quad (2.8)$$

where $\Im(a)$ stands for the imaginary part of the complex number a .

2.2 Pressure and temperature equations

Once we have estimated the absorbed energy Π_a , we may focus on the heating/pressure wave generation processes. Temperature rise and pressure are coupled. Note that most of the TAT models of the literature do not consider the heating process, which is often assumed to be *instantaneous*. This approximation can be justified in the photoacoustic case, where time scales are quite different: roughly speaking the lightning (and heating) process is much faster than the acoustic wave propagation. When the source is a radio-frequency one, this assumption may not be relevant any longer.

According to [10], the temperature and the pressure are determined by coupled equations which derive from the mass, momentum and energy conservation equations. Assuming that the variations in temperature and pressure are weak, the nonlinear effects are neglected and we obtain the following set of equations, for the temperature rise θ and the pressure perturbation p from the equilibrium steady state:

$$\begin{cases} \frac{\partial^2}{\partial t^2} (K_\theta p - \beta \theta)(t, x) = \operatorname{div}_x \left(\frac{1}{\rho} \nabla p \right)(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \frac{\partial}{\partial t} (\rho C_p \theta - \theta_0 \beta p)(t, x) = \operatorname{div}_x (\kappa \nabla \theta)(t, x) + \Pi_a(t, x), & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \end{cases}$$

where

- K_θ is the isothermal compressibility,
 - β is the volume thermal expansivity,
 - ρ is the mass density at steady state,
 - κ is the thermal conductivity,
 - θ_0 is the background temperature,
 - C_p is the isobar specific heat capacity (quantity of heat required to raise the temperature of a unit mass).
- All these parameter are assumed to be independent of the time t , but may depend on the position x , see Table 1 for some typical values of these parameters.

A combination of the above equations leads to

$$\left(\rho K_\theta - \frac{\theta_0 \beta^2}{C_p} \right) \frac{\partial^2 p}{\partial t^2} - \rho \operatorname{div}_x \left(\frac{1}{\rho} \nabla p \right) = \frac{\beta}{C_p} \frac{\partial \Pi_a}{\partial t} + \frac{\beta}{C_p} \frac{\partial}{\partial t} \operatorname{div}_x (\kappa \nabla \theta).$$

Let us introduce the acoustic wave velocity v_s which is defined by

$$\frac{1}{v_s^2} = \rho K_\theta - \frac{\theta_0 \beta^2}{C_p}$$

to obtain the thermoacoustic equation in heterogeneous media

$$\frac{1}{v_s^2} \frac{\partial^2 p}{\partial t^2} - \rho \operatorname{div}_x \left(\frac{1}{\rho} \nabla p \right) = \frac{\beta}{C_p} \frac{\partial \Pi_a}{\partial t} + \frac{\beta}{C_p} \frac{\partial}{\partial t} \operatorname{div}_x (\kappa \nabla \theta).$$

We further introduce the Grüneisen parameter Γ defined as

$$\Gamma = \frac{\beta v_s^2}{C_p}.$$

It is worth mentioning that v_s and the Grüneisen parameter Γ may be spatially heterogeneous. The coupled equations now read

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} - \rho v_s^2 \operatorname{div}_x \left(\frac{1}{\rho} \nabla p \right) - \Gamma \frac{\partial}{\partial t} \operatorname{div}_x (\kappa \nabla \theta) = \Gamma \frac{\partial \Pi_a}{\partial t}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \frac{\partial \theta}{\partial t} - \frac{1}{\rho C_p} \operatorname{div}_x (\kappa \nabla \theta) - \frac{\theta_0 \beta}{\rho C_p} \frac{\partial p}{\partial t} = \frac{\Pi_a}{\rho C_p}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3. \end{cases} \quad (2.9)$$

The initial conditions are such that there are no temperature or pressure differences before the absorption of the electromagnetic energy:

$$\theta(0, x) = 0, \quad p(0, x) = 0, \quad \frac{\partial p}{\partial t}(0, x) = 0, \quad x \in \mathbb{R}^3.$$

Without loss of generality, we reformulate equation (2.9) by introducing

$$P(t, x) = \int_0^t p(\tau, x) d\tau,$$

which corresponds to the integrated pressure. As $\frac{\partial P}{\partial t} = p$, integrating the first equation in (2.9) with respect to t gives

$$\begin{cases} \frac{\partial^2 P}{\partial t^2} - \rho v_s^2 \operatorname{div}_x \left(\frac{1}{\rho} \nabla P \right) - \Gamma \operatorname{div}_x (\kappa \nabla \theta) = \Gamma \Pi_a, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \frac{\partial \theta}{\partial t} - \frac{1}{\rho C_p} \operatorname{div}_x (\kappa \nabla \theta) - \frac{\theta_0 \beta}{\rho C_p} \frac{\partial^2 P}{\partial t^2} = \frac{\Pi_a}{\rho C_p}, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \end{cases} \quad (2.10)$$

with the initial conditions

$$\theta(0, x) = 0, \quad P(0, x) = 0, \quad \frac{\partial P}{\partial t}(0, x) = 0, \quad x \in \mathbb{R}^3.$$

2.3 Typical values of soft tissues

To be able to fully understand the time scale and evolution of the various physical quantities, some typical values of the various physical parameters are provided in Table 1. These typical values correspond to the ones of soft tissues that can be found when one considers a biomedical application, as for example breast cancer imaging [8, 11, 14, 23].

Parameter	Symbol	Value	Units
Permittivity of the vacuum	ϵ_0	$8.85 \cdot 10^{-12}$	F.m ⁻¹
Permeability of the vacuum	μ_0	$1.25 \cdot 10^{-6}$	H.m ⁻¹
Electric susceptibility (fat/glandular)	ξ	7.3 / 35.5	–
Conductivity (fat/glandular)	σ	0.23 / 1.34	S.m ⁻¹
Isobar specific heat capacity	C_p	$3.5 - 4 \times 10^3$	J.K ⁻¹ .kg ⁻¹
Isothermal compressibility	$K_\theta \sim \frac{1}{\rho v_s^2}$	5×10^{-10}	Pa ⁻¹
Volume thermal expansivity	β	4×10^{-4}	K ⁻¹
Mass density	ρ	1000	kg.m ⁻³
Thermal conductivity	κ	0.4	W.m ⁻¹ .K ⁻¹
Background temperature (20°C)	θ_0	293	K

Table 1. Approximated values of the physical parameters. The electric susceptibility and conductivity are measured at 3 GHz and at room temperature and correspond to fat and glandular tissues, respectively.

The acoustic transducers are typically wide band sensors with a central frequency of the order of 1 to 10 MHz, leading to an acoustic wavelength λ_a in the range of 0.1 to 1 mm (see [9, 27]). Meanwhile, the thermal diffusivity $D_\theta = \kappa/(\rho C_p)$ is approximately equal to $10^{-7} \text{ m}^2\text{s}^{-1}$. Hence, the thermal confinement threshold $(\lambda_a/2)^2/D_\theta$ is of the order of 0.05 to 5 s (see [34]). For high-resolution imaging, the thermal confinement duration is more or less ten times the duration of the illumination (which lasts few μs). Thus, in those cases, it seems rather difficult to neglect at first glance the coupling term between the thermal and acoustic effects.

3 Modelling of the inverse problem within the slowly time-varying envelope approximation

3.1 The infinite-dimensional model

As we have already seen, if the electromagnetic signal presents a slowly time-varying envelope, meaning that \hat{J}^{ex} takes the form (2.6), then \hat{E} and $\hat{\chi}$ admit frequency support in $W_c = [\omega_c - \Delta_c, \omega_c + \Delta_c]$ and

$$\hat{\chi}(\omega, x) \approx \hat{\chi}(\omega_c, x) \quad \text{for all } x \in \mathbb{R}^3, \omega \in W_c.$$

In the sequel, we denote $\hat{\chi}_c := \hat{\chi}(\omega_c, \cdot)$ and $\hat{\chi}_c^{\text{ex}} := \hat{\chi}^{\text{ex}}(\omega_c, \cdot)$. We recall that $\hat{\chi}_c - \hat{\chi}_c^{\text{ex}}$ has (spatial) compact support.

So, putting together equations (2.3), (2.8) and (2.10), the direct problem reads: Solve the system

$$\text{curl}_x \text{curl}_x \hat{E} - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c) \hat{E} = i\omega\mu_0 \hat{J}^{\text{ex}} \quad \text{in } W_c \times \mathbb{R}^3, \quad (3.1a)$$

$$\Pi_a(t, x) = \frac{\epsilon_0 \omega_c}{2\pi^2} \mathbb{J}(\hat{\chi}_c)(x) \left| \int_{W_c} \hat{E}(\omega, x) e^{-i\omega t} d\omega \right|^2 \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \quad (3.1b)$$

$$\frac{\partial^2 P}{\partial t^2} - \rho v_s^2 \text{div}_x \left(\frac{1}{\rho} \nabla P \right) - \Gamma \text{div}_x (\kappa \nabla \theta) = \Gamma \Pi_a \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \quad (3.1c)$$

$$\frac{\partial \theta}{\partial t} - \frac{1}{\rho C_p} \text{div}_x (\kappa \nabla \theta) - \frac{\theta_0 \beta}{\rho C_p} \frac{\partial^2 P}{\partial t^2} = \frac{\Pi_a}{\rho C_p} \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \quad (3.1d)$$

with initial conditions

$$\theta(0, x) = 0, P(0, x) = \frac{\partial P}{\partial t}(0, x) = 0 \quad \text{for all } x \in \mathbb{R}^3, \quad (3.2)$$

together with the radiation condition

$$\lim_{|x| \rightarrow \infty} (x \times \text{curl}_x \hat{E} - i \frac{\omega}{c_0} \sqrt{1 + \hat{\chi}_c^{\text{ex}}} |x| \hat{E}) = 0,$$

where the convergence should be uniformly in $x/|x|$, see (2.4); to obtain the Fourier transform of the electric field \hat{E} , the averaged absorbed energy Π_a , the temperature rise θ (supported in Ω) and the integrated pressure P .

Remark 3.1. There exist partial results for Maxwell's equations (3.1a). We know, for example that for a fixed frequency ω_c , if $\hat{\chi}_c \in L^\infty(\mathbb{C}^3)$ and $\hat{J}_c^{\text{ex}} \in L^2(\mathbb{C}^3)$ are compact supported (up to a constant function) in \mathbb{R}^3 , then problem (3.1a) has a unique solution in $H_{\text{loc}}(\text{curl}_x, \mathbb{R}^3)$, see [25], where

$$H_{\text{loc}}(\text{curl}_x, \mathbb{R}^3) = \{V : \mathbb{R}^3 \rightarrow \mathbb{C}^3 : V|_B \in H(\text{curl}_x, B) \text{ for all balls } B \subseteq \mathbb{R}^3\}$$

and $H(\text{curl}_x, B) = \{V \in L^2(B) : \text{curl}_x V \in L^2(B)\}$.

There are no straightforward existence and/or uniqueness results for the coupled system (3.1c)–(3.1d). If we fix one of the variables θ or P , the equations are classical and we get many existence and regularity results under assumptions on the *fixed* unknown. However, the coupling imposes to use specific methods (fixed point or diagonalization methods) that we do not want to check in the present paper. This will be addressed in a forthcoming work where rigorous mathematical analysis will be performed.

Remark 3.2. In the above derivation, we considered the total (electromagnetic) field. We did not use the *perturbed* field (or scattered field) E_s which is directly linked to $\hat{\chi} - \hat{\chi}^{\text{ex}}$ since the dissipation Π_a as a function of E_s is much more complicated.

We are now interested in the inverse problem: given a known source J^{ex} and integrated pressure measurements P_{obs} on a set $(0, T) \times \mathcal{W}$, recover $\hat{\chi}_c$. Here, the set \mathcal{W} denotes the area of the transducers. Following [7] we define the functional \mathcal{J} by

$$\mathcal{J}(\hat{\chi}_c) = \mathcal{F}(\hat{\chi}_c) + f(\hat{\chi}_c).$$

Here $f(\hat{\chi}_c)$ stands for a regularizing term and \mathcal{F} is a least square functional with respect to the measured integrated pressure data. We set

$$\mathcal{F}(\hat{\chi}_c) = \frac{1}{2} \int_{(0, T) \times \mathcal{W}} (P[\hat{\chi}_c](t, x) - P_{\text{obs}}(t, x))^2 dx dt,$$

where P_{obs} is the measured pressure on $(0, T) \times \mathcal{W}$ when the source signal is J^{ex} . Here $P[\hat{\chi}_c]$ is the solution of equations (3.1c)–(3.1d) with (3.2). We may define f as

$$f(\hat{\chi}_c) = \frac{\alpha_0}{2} \int_{W_c \times \mathbb{R}^3} |\hat{\chi} - \hat{\chi}^{\text{ex}}|^2(\omega, x) d\omega dx = \frac{2\Delta_c \alpha_0}{2} \int_{\mathbb{R}^3} |\hat{\chi}_c - \hat{\chi}_c^{\text{ex}}|^2(x) dx,$$

where $\alpha_0 > 0$, for example. Then the minimization problem reads

$$\min_{\hat{\chi}_c \in \mathcal{U}_{\text{ad}}} \mathcal{J}(\hat{\chi}_c), \quad (3.3)$$

where \mathcal{U}_{ad} is a closed set of admissible functions $\hat{\chi}_c$ such that $\hat{\chi}_c = \hat{\chi}_c^{\text{ex}}$ outside Ω .

In the sequel, we set $\alpha = 2\Delta_c \alpha_0$ so that

$$f(\hat{\chi}_c) = \frac{\alpha}{2} \int_{\mathbb{R}^3} |\hat{\chi}_c - \hat{\chi}_c^{\text{ex}}|^2(x) dx.$$

3.2 Formal formulation of the inverse problem

The above formulation is of course quite formal. The study of problem (3.3) in an infinite-dimensional setting is challenging and deserves to be precisely studied. Indeed, the operator $\mathcal{T} : \hat{\chi}_c \mapsto P[\hat{\chi}_c]$ is not easy to handle. In particular, its continuity properties strongly depend on the functional framework and regularity properties of the solutions of the strongly coupled system (3.1).

We now give a *formal* analysis of the inverse problem to present the whole reconstruction process. This may be completely justified in a finite-dimensional (discrete) setting. It is not our concern to enter in the discretization details and we keep the same notations as in the infinite-dimensional setting. However, as we have numerics in mind, we have replaced \mathbb{R}_+ by $(0, T)$ with some terminal time T large enough, and \mathbb{R}^3 by some large ball \mathcal{B} . We use a Dirichlet boundary condition on $\partial\mathcal{B}$ (recall that T is chosen large enough to allow it). The discretization process is a challenging issue that we do not address in the present paper.

The direct problem now reads

$$\text{curl}_x \text{curl}_x \hat{E} - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c) \hat{E} = i\omega\mu_0 \hat{J}^{\text{ex}} \quad \text{in } W_c \times \mathcal{B}, \quad (3.4a)$$

$$\Pi_a(t, x) = \frac{\epsilon_0 \omega_c}{2\pi^2} \mathfrak{I}(\hat{\chi}_c(x)) \left| \int_{W_c} \hat{E}(\omega, x) e^{-i\omega t} d\omega \right|^2 \quad \text{in } (0, T) \times \mathcal{B}, \quad (3.4b)$$

$$\frac{\partial^2 P}{\partial t^2} - \rho v_s^2 \text{div}_x \left(\frac{1}{\rho} \nabla P \right) - \Gamma \text{div}_x (\kappa \nabla \theta) = \Gamma \Pi_a \quad \text{in } (0, T) \times \mathcal{B}, \quad (3.4c)$$

$$\frac{\partial \theta}{\partial t} - \frac{1}{\rho C_p} \text{div}_x (\kappa \nabla \theta) - \frac{\theta_0 \beta}{\rho C_p} \frac{\partial^2 P}{\partial t^2} = \frac{\Pi_a}{\rho C_p} \quad \text{in } (0, T) \times \mathcal{B}, \quad (3.4d)$$

with initial conditions

$$\theta(0, x) = 0, \quad P(0, x) = \frac{\partial P}{\partial t}(0, x) = 0 \quad \text{in } \mathcal{B},$$

and boundary conditions

$$\hat{E}(\omega, x) = 0, \quad (\omega, x) \in W_c \times \partial\mathcal{B}, \quad P(t, x) = 0, \quad \frac{\partial P}{\partial t}(t, x) = 0, \quad \theta(t, x) = 0, \quad (t, x) \in (0, T) \times \partial\mathcal{B}.$$

Remark 3.3. We decided to set a Dirichlet boundary condition for the \hat{E} field. This is the case for example if either

- $\hat{\chi}^{\text{ex}}$ is lossless and the ball \mathcal{B} is very large,
- $\hat{\chi}^{\text{ex}}$ is lossless and there is a metallic casing around \mathcal{B} , or
- $\hat{\chi}^{\text{ex}}$ is lossy and the waves will be automatically damped.

Another possibility could be to use a Robin-type boundary, which can be seen as a first-order approximation of the Silver–Müller radiation boundary condition on a bounded surface:

$$n \times \text{curl}_x \hat{E} - i \frac{\omega}{c_0} \sqrt{1 + \hat{\chi}^{\text{ex}}} n \times n \times \hat{E} = 0,$$

where n is the unit outward normal vector on $\partial\mathcal{B}$.

Note that the issue is the same for the acoustic pressure, but as we are looking for p (or P) which only correspond to the variation of the pressure with respect to the initial one, it seems rather natural to assume that $P = 0$ on the boundary.

Theorem 3.1. Assume \mathcal{U}_{ad} is bounded or $\alpha > 0$. Then the discrete minimization problem has at least a solution.

Proof. This is clear since either the admissible set \mathcal{U}_{ad} is bounded or \mathcal{J} is coercive. Moreover, the continuity properties are fulfilled in the discrete framework. \square

Uniqueness is a challenging problem, since the cost functional is not convex (because of the fitting data term). Nevertheless, any solution $\hat{\chi}_c^*$ satisfies

$$\nabla \mathcal{J}(\hat{\chi}_c^*) \cdot (\hat{\chi}_c - \hat{\chi}_c^*) \geq 0 \quad \text{for all } \hat{\chi}_c \in \mathcal{U}_{\text{ad}}.$$

The next subsections are devoted to the computation of $\nabla \mathcal{J}(\hat{\chi}_c^*)$ and the setting of a necessary optimality system.

We first start to compute $\nabla \mathcal{F}(\hat{\chi}_c^*) \cdot \hat{\chi}_c$ before tackling the term in $\nabla \mathcal{J}(\hat{\chi}_c^*) \cdot \hat{\chi}_c$. In the sequel $\hat{\chi}_c^*$ denotes an optimal solution and \hat{E}^* , P^* and θ^* are the corresponding solutions of (the discrete) system (3.4). We previously adopted the notation \cdot for the \mathbb{R}^3 inner product. However, in the sequel we omit this dot, to make computations more readable. More precisely, if $A = (A_k)_{1 \leq k \leq 3}$ and $B = (B_k)_{1 \leq k \leq 3}$ are vector-valued functions from \mathbb{R}^3 to \mathbb{R}^3 , we write $A(x)B(x)$ instead of $A(x) \cdot B(x) (= \sum_{k=1}^3 A_k(x)B_k(x))$.

3.3 The (discrete) pressure-temperature system

The coupling between equations (3.4a)–(3.4b) and (3.4c)–(3.4d) is done via the Π_a term. Let $\hat{\chi}_c \in \mathcal{U}_{\text{ad}}$ and denote by $\dot{\Pi}_a$ the derivative of Π_a at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$:

$$\dot{\Pi}_a \cdot \hat{\chi}_c := \lim_{\rho \rightarrow 0^+} \frac{\Pi_a(\hat{\chi}_c^* + \rho \hat{\chi}_c) - \Pi_a(\hat{\chi}_c^*)}{\rho}.$$

Let us denote similarly \dot{P} and $\dot{\theta}$ the derivatives of P and θ at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$. So

$$\nabla \mathcal{F}(\hat{\chi}_c^*) \cdot \hat{\chi}_c = \int_{(0, T) \times \mathcal{W}} (P^* - P_{\text{obs}}) \dot{P} \, dt \, dx,$$

where $P^* = P[\hat{\chi}_c^*]$ and $\theta^* = \theta[\hat{\chi}_c^*]$. The equations satisfied by \dot{P} and $\dot{\theta}$ are

$$\begin{aligned} \frac{\partial^2 \dot{P}}{\partial t^2} - \rho v_s^2 \text{div}_x \left(\frac{1}{\rho} \nabla \dot{P} \right) - \Gamma \text{div}_x (\kappa \nabla \dot{\theta}) &= \Gamma \dot{\Pi}_a \quad \text{in } (0, T) \times \mathcal{B}, \\ \rho C_p \frac{\partial \dot{\theta}}{\partial t} - \text{div}_x (\kappa \nabla \dot{\theta}) - \theta_0 \beta \frac{\partial^2 \dot{P}}{\partial t^2} &= \dot{\Pi}_a \quad \text{in } (0, T) \times \mathcal{B}, \end{aligned}$$

with initial and boundary conditions

$$\dot{\theta}(0, x) = 0, \quad \dot{P}(0, x) = 0, \quad \frac{\partial \dot{P}}{\partial t}(0, x) = 0, \quad x \in \mathcal{B},$$

$$\dot{P}(t, x) = 0, \quad \frac{\partial \dot{P}}{\partial t}(t, x) = 0, \quad \dot{\theta}(t, x) = 0, \quad (t, x) \in (0, T) \times \partial \mathcal{B}.$$

Let us define the adjoint system for (P^*, θ^*) . We consider the following system for q^* and ζ^* :

$$\frac{\partial^2 q^*}{\partial t^2} - \theta_0 \beta \frac{\partial^2 \zeta^*}{\partial t^2} - \operatorname{div}_x \left(\frac{1}{\rho} \nabla (\rho v_s^2 q^*) \right) = (P^* - P_{\text{obs}}) \mathbf{1}_{(0, T) \times \mathcal{W}} \quad \text{in } (0, T) \times \mathcal{B}, \quad (3.5a)$$

$$-\rho C_p \frac{\partial \zeta^*}{\partial t} - \operatorname{div}_x (\kappa \nabla (\Gamma q^*)) - \operatorname{div}_x (\kappa \nabla \zeta^*) = 0 \quad \text{in } (0, T) \times \mathcal{B}, \quad (3.5b)$$

with initial and boundary conditions

$$\zeta^*(T, x) = 0, \quad q^*(T, x) = 0, \quad \frac{\partial q^*}{\partial t}(T, x) = 0, \quad x \in \mathcal{B}, \quad (3.6a)$$

$$q^*(t, x) = 0, \quad \frac{\partial q^*}{\partial t}(t, x) = 0, \quad \zeta^*(t, x) = 0, \quad (t, x) \in (0, T) \times \partial \mathcal{B}. \quad (3.6b)$$

Then we get

Lemma 3.1. *The gradient of \mathcal{F} at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$ is given by*

$$\nabla \mathcal{F}(\hat{\chi}_c^*) \cdot \hat{\chi}_c = \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*) \dot{\Pi}_a \, dt \, dx.$$

where (q^*, ζ^*) are defined by the adjoint equations (3.5)–(3.6), and $\dot{\Pi}_a$ is the derivative of Π_a at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$.

Proof. Using integration by parts, we obtain

$$\begin{aligned} \nabla \mathcal{F}(\hat{\chi}_c^*) \cdot \hat{\chi}_c &= \int_{(0, T) \times \mathcal{W}} (P^* - P_{\text{obs}}) \dot{P} \, dt \, dx \\ &= \int_{(0, T) \times \mathcal{B}} (P^* - P^{\text{obs}}) \dot{P} \mathbf{1}_{(0, T) \times \mathcal{W}} \, dt \, dx \\ &= \int_{(0, T) \times \mathcal{B}} \left[\frac{\partial^2 q^*}{\partial t^2} - \theta_0 \beta \frac{\partial^2 \zeta^*}{\partial t^2} - \operatorname{div}_x \left(\frac{1}{\rho} \nabla (\rho v_s^2 q^*) \right) \right] \dot{P} \, dt \, dx \\ &\quad + \underbrace{\int_{(0, T) \times \mathcal{B}} \left(-\rho C_p \frac{\partial \zeta^*}{\partial t} - \operatorname{div}_x (\kappa \nabla (\Gamma q^*)) - \operatorname{div}_x (\kappa \nabla \zeta^*) \right) \dot{\theta} \, dt \, dx}_{=0} \\ &= \int_{(0, T) \times \mathcal{B}} \left\{ \left[\frac{\partial^2 \dot{P}}{\partial t^2} - \rho v_s^2 \operatorname{div}_x \left(\frac{1}{\rho} \nabla \dot{P} \right) \right] q^* - \theta_0 \beta \frac{\partial^2 \dot{P}}{\partial t^2} \zeta^* \right\} \, dt \, dx \\ &\quad + \int_{(0, T) \times \mathcal{B}} \left\{ \left(\rho C_p \frac{\partial \dot{\theta}}{\partial t} - \operatorname{div}_x (\kappa \nabla \dot{\theta}) \right) \zeta^* - \Gamma \operatorname{div}_x (\kappa \nabla \dot{\theta}) q^* \right\} \, dt \, dx \\ &= \int_{(0, T) \times \mathcal{B}} \left[\frac{\partial^2 \dot{P}}{\partial t^2} - \rho v_s^2 \operatorname{div}_x \left(\frac{1}{\rho} \nabla \dot{P} \right) - \Gamma \operatorname{div}_x (\kappa \nabla \dot{\theta}) \right] q^* \, dt \, dx \\ &\quad + \int_{(0, T) \times \mathcal{B}} \left(\rho C_p \frac{\partial \dot{\theta}}{\partial t} - \operatorname{div}_x (\kappa \nabla \dot{\theta}) - \theta_0 \beta \frac{\partial^2 \dot{P}}{\partial t^2} \right) \zeta^* \, dt \, dx \\ &= \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*) \dot{\Pi}_a \, dt \, dx. \quad \square \end{aligned}$$

Now, we have to estimate $\dot{\Pi}_a$ introducing an adjoint equation for the electromagnetic part.

3.4 (Discrete) Maxwell's equations

Let us now define the adjoint state Φ^* associated to E^* . To simplify the notation, we introduce the quantities

$$\mathcal{E}^*(t, x) = \int_{W_c} \hat{E}^*(\omega, x) e^{-i\omega t} d\omega \quad \text{and} \quad \dot{\mathcal{E}}(t, x) = \int_{W_c} \dot{\hat{E}}(\omega, x) e^{-i\omega t} d\omega. \quad (3.7)$$

Then we set Φ^* to be the solution of the equation

$$\text{curl}_x \text{curl}_x \Phi^*(\omega, \cdot) - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c^*) \Phi^*(\omega, \cdot) = \mathbb{J}(\hat{\chi}_c^*) \left(\int_0^T (\Gamma q^* + \zeta^*)(t, \cdot) \overline{\mathcal{E}^*(t, \cdot)} e^{-i\omega t} dt \right) \quad \text{in } W_c \times \mathcal{B} \quad (3.8)$$

with the homogeneous Dirichlet boundary conditions

$$\Phi^*(\omega, x) = 0, \quad (\omega, x) \in W_c \times \partial\mathcal{B}.$$

Proposition 3.1. *The gradient of \mathcal{F} at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$ is given by*

$$\begin{aligned} \nabla \mathcal{F}(\hat{\chi}_c^*) \cdot \hat{\chi}_c &= \frac{\epsilon_0 \omega_c}{2\pi^2} \int_{\mathcal{B}} \left\{ \left(\int_0^T (\Gamma q^* + \zeta^*)(t, x) |\mathcal{E}^*(t, x)|^2 dt \right) \mathbb{J}(\hat{\chi}_c(x)) \right. \\ &\quad + 2 \left(\Re \int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) d\omega \right) \Re(\hat{\chi}_c(x)) \\ &\quad \left. - 2 \left(\Im \int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) d\omega \right) \Im(\hat{\chi}_c(x)) \right\} dx, \end{aligned} \quad (3.9)$$

where (q^*, ζ^*) are defined by the adjoint equations (3.5)–(3.6), and Φ^* by (3.8).

Proof. Let us denote by $\dot{\hat{E}}$ the derivative of \hat{E} at $\hat{\chi}_c^*$ in the direction $\hat{\chi}_c$. The equations satisfied by $\dot{\Pi}_a$ and $\dot{\hat{E}}$ are

$$\begin{aligned} \text{curl}_x \text{curl}_x \dot{\hat{E}} - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c^*) \dot{\hat{E}} &= \frac{\omega^2 \dot{\hat{E}}^*}{c_0^2} \hat{\chi}_c \quad \text{in } W_c \times \mathcal{B}, \\ \dot{\Pi}_a(t, x) &= \frac{\epsilon_0 \omega_c}{\pi^2} \left(\frac{1}{2} \mathbb{J}(\hat{\chi}_c(x)) \left| \int_{W_c} \hat{E}^*(\omega, x) e^{-i\omega t} d\omega \right|^2 \right. \\ &\quad \left. + \mathbb{J}(\hat{\chi}_c^*(x)) \Re \left[\int_{W_c} \dot{\hat{E}}(\omega, x) e^{-i\omega t} d\omega \int_{W_c} \overline{\hat{E}^*(\omega, x) e^{-i\omega t}} d\omega \right] \right) \quad \text{in } (0, T) \times \mathcal{B}. \end{aligned}$$

Equivalently, with the notation (3.7),

$$\dot{\Pi}_a(t, x) = \frac{\epsilon_0 \omega_c}{2\pi^2} \mathbb{J}(\hat{\chi}_c(x)) |\mathcal{E}^*(t, x)|^2 + \frac{\epsilon_0 \omega_c}{\pi^2} \mathbb{J}(\hat{\chi}_c^*(x)) \Re[\dot{\mathcal{E}}(t, x) \overline{\mathcal{E}^*(t, x)}] \quad \text{in } (0, T) \times \mathcal{B}.$$

So

$$\begin{aligned} \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*) \dot{\Pi}_a(t, x) dt dx &= \frac{\epsilon_0 \omega_c}{2\pi^2} \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*) \mathbb{J}(\hat{\chi}_c(x)) |\mathcal{E}^*(t, x)|^2 dt dx \\ &\quad + \frac{\epsilon_0 \omega_c}{\pi^2} \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*) \mathbb{J}(\hat{\chi}_c^*(x)) \Re(\dot{\mathcal{E}}(t, x) \overline{\mathcal{E}^*(t, x)}) dt dx. \end{aligned}$$

As the $\text{curl}_x \text{curl}_x$ operator is self-adjoint, performing integration by parts gives

$$\begin{aligned} &\int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*)(t, x) \mathbb{J}(\hat{\chi}_c^*(x)) \dot{\mathcal{E}}(t, x) \overline{\mathcal{E}^*(t, x)} dx dt \\ &= \int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*)(t, x) \mathbb{J}(\hat{\chi}_c^*(x)) \left(\int_{W_c} \dot{\hat{E}}(\omega, x) e^{-i\omega t} d\omega \right) \overline{\mathcal{E}^*(t, x)} dx dt \\ &= \int_{W_c \times \mathcal{B}} \mathbb{J}(\hat{\chi}_c^*(x)) \left(\int_0^T (\Gamma q^* + \zeta^*)(t, x) \overline{\mathcal{E}^*(t, x)} e^{-i\omega t} dt \right) \dot{\hat{E}}(\omega, x) d\omega dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{W_c \times \mathcal{B}} \left(\operatorname{curl}_x \operatorname{curl}_x \Phi^* - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c^*) \Phi^* \right) (\omega, x) \dot{E}(\omega, x) \, d\omega \, dx \\
 &= \int_{W_c \times \mathcal{B}} \left(\operatorname{curl}_x \operatorname{curl}_x \dot{E} - \frac{\omega^2}{c_0^2} (1 + \hat{\chi}_c^*) \dot{E} \right) (\omega, x) \Phi^*(\omega, x) \, d\omega \, dx \\
 &= \int_{W_c \times \mathcal{B}} \frac{\omega^2 \hat{E}^* \Phi^*}{c_0^2} (\omega, x) \hat{\chi}_c(x) \, d\omega \, dx \\
 &= \int_{\mathcal{B}} \left(\int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \hat{\chi}_c(x) \, dx.
 \end{aligned}$$

This gives the result since

$$\begin{aligned}
 &\int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*)(t, x) \mathbb{J}(\hat{\chi}_c^*(x)) \Re(\dot{E}(t, x) \overline{\mathcal{E}^*(t, x)}) \, dt \, dx \\
 &= \Re \left(\int_{(0, T) \times \mathcal{B}} (\Gamma q^* + \zeta^*)(t, x) \mathbb{J}(\hat{\chi}_c^*(x)) \dot{E}(t, x) \overline{\mathcal{E}^*(t, x)} \, dt \, dx \right) \\
 &= \Re \left(\int_{\mathcal{B}} \left(\int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \hat{\chi}_c(x) \, dx \right) \\
 &= \int_{\mathcal{B}} \Re \left(\int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \Re(\hat{\chi}_c(x)) \, dx \\
 &\quad - \int_{\mathcal{B}} \mathbb{J} \left(\int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \mathbb{J}(\hat{\chi}_c(x)) \, dx. \quad \square
 \end{aligned}$$

3.5 The optimality system

For the regularization term f , we find that

$$\nabla f(\hat{\chi}_c^*) \cdot \hat{\chi}_c = \alpha \Re \left(\int_{\mathcal{B}} \overline{(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}})}(x) \hat{\chi}_c(x) \, dx \right) = \alpha \int_{\mathcal{B}} (\Re(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}}) \Re(\hat{\chi}_c) + \mathbb{J}(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}}) \mathbb{J}(\hat{\chi}_c))(x) \, dx. \quad (3.10)$$

Thus, putting together the expressions (3.9) and (3.10), we have for every $\hat{\chi}_c \in \mathcal{U}_{\text{ad}}$,

$$\begin{aligned}
 \nabla \mathcal{J}(\hat{\chi}_c^*) \cdot (\hat{\chi}_c - \hat{\chi}_c^*) &= \frac{\epsilon_0 \omega_c}{2\pi^2} \int_{\mathcal{B}} \left\{ \left(\int_0^T (\Gamma q^* + \zeta^*)(t, x) |\mathcal{E}^*(t, x)|^2 \, dt \right) \mathbb{J}(\hat{\chi}_c - \hat{\chi}_c^*)(x) \right. \\
 &\quad + 2 \left(\Re \int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \Re(\hat{\chi}_c - \hat{\chi}_c^*)(x) \\
 &\quad \left. - 2 \left(\mathbb{J} \int_{W_c} \frac{\omega^2}{c_0^2} \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega \right) \mathbb{J}(\hat{\chi}_c - \hat{\chi}_c^*)(x) \right\} \, dx \\
 &\quad + \alpha \int_{\mathcal{B}} [\Re(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}}) \Re(\hat{\chi}_c - \hat{\chi}_c^*) + \mathbb{J}(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}}) \mathbb{J}(\hat{\chi}_c - \hat{\chi}_c^*)](x) \, dx \geq 0. \quad (3.11)
 \end{aligned}$$

Decoupling with respect to real and imaginary parts, we get for any (v_r, v_i) real-valued functions such that $v_r + i\mathbb{J}(\hat{\chi}_c^*) = \hat{\chi}_c \in \mathcal{U}_{\text{ad}}$ and $\Re(\hat{\chi}_c^*) + iv_i = \hat{\chi}_c \in \mathcal{U}_{\text{ad}}$,

$$\int_{\mathcal{B}} \Re \left(\frac{\epsilon_0 \omega_c}{c_0^2 \pi^2} \int_{W_c} \omega^2 \hat{E}^*(\omega, x) \Phi^*(\omega, x) \, d\omega + \alpha (\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}})(x) \right) (v_r - \Re(\hat{\chi}_c^*)) \, dx \geq 0 \quad (3.12a)$$

and

$$\int_{\mathcal{B}} \left\{ \frac{\epsilon_0 \omega_c}{2\pi^2} \left(\int_0^T (\Gamma q^* + \zeta^*)(t, x) |\mathcal{E}^*(t, x)|^2 dt \right) - \mathbb{J} \left(\frac{\epsilon_0 \omega_c}{c_0^2 \pi^2} \int_{W_c} \omega^2 \hat{E}^*(\omega, x) \Phi^*(\omega, x) d\omega - \alpha(\hat{\chi}_c^* - \hat{\chi}_c^{\text{ex}})(x) \right) \right\} (v_i - \mathbb{J}(\hat{\chi}_c^*))(x) dx \geq 0. \quad (3.12b)$$

We may now give first order necessary conditions to get a solution of the discretized problem (3.3).

Theorem 3.2. *Let χ_c^* be an optimal solution of the minimization problem (3.3). Then the following optimality system is satisfied:*

- *State equations: (3.4) with the corresponding initial and boundary conditions.*
- *Adjoint equations: (3.5)–(3.6) for pressure and temperature, and (3.8) for electrical field.*
- *Projection equations: (3.11) (or equivalently (3.12)).*

4 Conclusion

We have presented what we consider as a complete modelling of the thermoacoustic process with microwave sources. Since the complete model is quite complex, we focused on the very case where the source signal presents a slowly time-varying envelope which is practically relevant. We have presented the inverse and the method to solve it in a formal way. Next steps are the complete theoretical mathematical analysis in an infinite-dimensional framework and the numerical experimentation (solving the optimality system). This is challenging since there are many open problems, in particular in terms of the overall numerical burden that this coupling formalism will generate. However, the electromagnetic system (in the Fourier space) and the pressure-temperature are slightly coupled. The next issue will be the mathematical study of the pressure-temperature system together with (3.5)–(3.6) and the identification of Π_a in a first step.

Funding: This work is supported by ANR (AVENTURES – ANR-12-BLAN-BS01-0001-01) and Partenariat Hubert Curien AMADEUS, OeAD WTZ FR14/2013.

References

- [1] H. Ammari, E. Bossy, V. Jugnon and H. Kang, Reconstruction of the optical absorption coefficient of a small absorber from the absorbed energy density, *SIAM J. Appl. Math.* **71** (2011), no. 3, 676–693.
- [2] H. Ammari, J. Garnier, W. Jing and L. H. Nguyen, Quantitative thermo-acoustic imaging: An exact reconstruction formula, *J. Differential Equations* **254** (2013), 1375–1395.
- [3] G. Bal, A. Jollivet and V. Jugnon, Inverse transport theory of photoacoustics, *Inverse Problems* **26** (2010), no. 2, Article ID 025011.
- [4] G. Bal and K. Ren, Multi-source quantitative photoacoustic tomography in a diffusive regime, *Inverse Problems* **27** (2011), Article ID 075003.
- [5] G. Bal, K. Ren, G. Uhlmann and T. Zhou, Quantitative thermo-acoustics and related problems, *Inverse Problems* **27** (2011), no. 5, Article ID 055007.
- [6] G. Bal and T. Zhou, Hybrid inverse problems for a system of Maxwell's equations, *Inverse Problems* **30** (2014), no. 5, Article ID 055013.
- [7] M. Bergounioux, X. Bonnefond, T. Haberkorn and Y. Privat, An optimal control problem in photoacoustic tomography, *Math. Models Methods Appl. Sci.* **24** (2014), no. 14, 2943–48.
- [8] M. J. Burfeindt, T. J. Colgan, R. O. Mays, J. D. Shea, N. Behdad, B. D. V. Veen and S. C. Hagness, MRI-derived 3D-printed breast phantom for microwave breast imaging validation, *IEEE Antennas and Wireless Propagation Lett.* **11** (2012), 1610–1613.
- [9] G. Chen, X. Wang and Q. Liu, Microwave-induced thermo-acoustic tomography system using TRM-PSTD technique, *PIER-B* **48** (2013), 43–59.

- [10] B. Cox and P. Beard, Modeling photoacoustic propagation in tissue using k -space techniques, in: *Photoacoustic Imaging and Spectroscopy*, CRC Press, Boca Raton (2009), 25–34.
- [11] F. Duck, *Physical Properties of Tissue: A Comprehensive Reference Book*, Institution of Physics & Engineering in Medicine & Biology, York, 2012.
- [12] P. Elbau, L. Mindrinos and O. Scherzer, Inverse problems of combined photoacoustic and optical coherence tomography, *Math. Methods Appl. Sci.* (2016), DOI 10.1002/mma.3915.
- [13] D. Fallon, L. Yan, G. W. Hanson and S. K. Patch, RF testbed for thermoacoustic tomography, *Rev. Sci. Instrument* **80** (2009), Article ID 064301.
- [14] S. Gabriel, R. Lau and C. Gabriel, The dielectric properties of biological tissues. II. Measurements in the frequency range 10 Hz to 20 GHz, *Phys. Med. Biol.* **41** (1996), no. 11, 2251–2269.
- [15] H. Gao, H. Zhao and S. Osher, Bregman methods in quantitative photoacoustic tomography, Technical Report 10-42, University of California, Los Angeles, 2010.
- [16] M. Haltmeier, L. Neumann and S. Rabanser, Single-stage reconstruction algorithm for quantitative photoacoustic tomography, *Inverse Problems* **31** (2015), no. 6, Article ID 065005.
- [17] L. Huang, L. Yao, L. Liu, J. Rong and H. Jiang, Quantitative thermoacoustic tomography: Recovery of conductivity maps of heterogeneous media, *Appl. Phys. Lett.* **101** (2012), no. 24, Article ID 244106.
- [18] J. Jackson, *Classical Electrodynamics*, Wiley, New York, 1998.
- [19] R. Kruger, W. Kiser, K. Miller and H. Reynolds, Thermoacoustic CT: Imaging principles, *Proc. SPIE* **3916** (2000), DOI 10.1117/12.386316.
- [20] R. A. Kruger, W. L. Kiser, D. R. Reinecke, G. A. Kruger and R. L. Eisenhart, Thermoacoustic computed tomography of the breast at 434 MHz, *IEEE MTT-S Internat. Microw. Sympos. Digest* **2** (1999), 591–595.
- [21] P. Kuchment and L. Kunyansky, Mathematics of thermoacoustic tomography, *European J. Appl. Math.* **19** (2008), 191–224.
- [22] P. Kuchment and L. Kunyansky, Mathematics of photoacoustic and thermoacoustic tomography, in: *Handbook of Mathematical Methods in Imaging*, Springer, New York (2015), 1117–1167.
- [23] M. Lazebnik, L. McCartney, D. Popovic, C. Watkins, M. Lindstrom, J. Harter, S. Sewall, A. Magliocco, J. Booske, M. Okoniewski and S. Hagness, A large-scale study of the ultrawideband microwave dielectric properties of normal breast tissue obtained from reduction surgeries, *Phys. Med. Biol.* **52** (2007), 2637–2656.
- [24] C. Li, M. Pramanik, G. Ku and L. Wang, Image distortion in thermoacoustic tomography caused by microwave diffraction, *Phys. Rev. E* **77** (2008), Article ID 31923.
- [25] P. Monk, *Finite Element Methods for Maxwell's Equations*, Oxford University Press, Oxford, 2003.
- [26] W. Naetar and O. Scherzer, Quantitative photoacoustic tomography with piecewise constant material parameters, *SIAM J. Imaging Sci.* **7** (2014), no. 3, 1755–1774.
- [27] L. Nie, D. Xing, D. Yang, L. Zeng and Q. Zhou, Detection of foreign body using fast thermoacoustic tomography with a multielement linear transducer array, *Appl. Phys. Lett.* **90** (2007), Article ID 174109.
- [28] S. K. Patch and O. Scherzer, Photo- and thermo-acoustic imaging introduction, *Inverse Problems* **23** (2007), no. 6, S1–S10.
- [29] M. Pramanik, G. Ku, C. Li and L. V. Wang, Design and evaluation of a novel breast cancer detection system combining both thermoacoustic (ta) and photoacoustic (pa) tomography, *Med. Phys.* **35** (2008), no. 6, 2218–2223.
- [30] T. Saratoon, T. Tarvainen, B. Cox and S. Arridge, A gradient-based method for quantitative photoacoustic tomography using the radiative transfer equation, *Inverse Problems* **29** (2013), Article ID 075006.
- [31] P. Shao, T. Harrison and R. J. Zemp, Iterative algorithm for multiple illumination photoacoustic tomography (mipat) using ultrasound channel data, *Biomed. Opt. Express.* **3** (2012), 3240–3249.
- [32] N. Song, D. C. and A. Da Silva, Considering sources and detectors distributions for quantitative photoacoustic tomography (qpat), *Biomed. Opt. Express.* **5** (2014), 3960–3974.
- [33] K. Wang and M. A. Anastasio, Photoacoustic and thermoacoustic tomography: Image formation principles, in: *Handbook of Mathematical Methods in Imaging*, Springer, New York (2015), 1081–1116.
- [34] L. Wang, *Photoacoustic Imaging and Spectroscopy*, Optical Sci. Eng. 144, CRC Press, Boca Raton, 2009.
- [35] L. Wang, X. Zhao, H. Sun and G. Ku, Microwave-induced acoustic imaging of biological tissues, *Rev. Sci. Instrum.* **70** (1999), no. 9, 3744–3748.
- [36] X. Wang, D. R. Bauer, R. Witte and H. Xin, Microwave-induced thermoacoustic imaging model for potential breast cancer detection, *IEEE Trans. Biomed. Eng.* **59** (2012), no. 10, 2782–2791.
- [37] Z. Yuan and H. Jiang, Quantitative photoacoustic tomography, *Philos. Trans. R. Soc. Math. Phys. Eng. Sci.* **367** (2009), 3043–54.
- [38] R. J. Zemp, Quantitative photoacoustic tomography with multiple optical sources, *Appl. Opt.* **49** (2010), no. 18, 3566–3572.