Exercise Sheet 3

1. Compute the QR-decomposition of the matrix
$$A = \begin{bmatrix} 3 & 7 \\ 0 & 12 \\ 4 & 1 \end{bmatrix}.$$

2. Consider the function $f(x) = x^4$. Find the Lagrange-polynomial that interpolates the function f at the points $x_0 = -1,\ x_1 = 0$ and $x_2 = 2$.

3. Consider the sums
$$\sum_{k=0}^{n-1} \cos(j\, t_k)\sin(\hat{j}\, t_k) \quad \text{and} \quad \sum_{k=0}^{n-1} \sin(j\, t_k)\sin(\hat{j}\, t_k),$$
where $t_k = k\frac{2\pi}{n}$ are the grid points. Using the formulas
$$\cos(j\, t_k)\sin(\hat{j}\, t_k) = \frac{1}{2}\operatorname{Im}\left\{e^{i(j+\hat{j})t_k} - e^{i(j-\hat{j})t_k}\right\}$$
and
$$\sin(j\, t_k)\sin(\hat{j}\, t_k) = \frac{1}{2}\operatorname{Re}\left\{e^{i(j-\hat{j})t_k} - e^{i(j+\hat{j})t_k}\right\},$$
calculate the above sums for all the possible values of $j, \hat{j} \in \left\{0, 1, ..., \frac{n-1}{2}\right\}$ following the same procedure as in the lecture notes for the relevant sum.

4. If p_i is a linear polynomial of the form
$$p_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i), \quad i = 0, ..., n$$
then $p_i(x) = s(x)$ for $x \in [x_i, x_{i+1}]$, where $s \in S_{1,\Delta}$ is the linear spline for the grid $\Delta = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of the interval $[a, b]$. Let $f(x) = x^2,\ x \in [0, 3]$. Approximate the function f at the nodal points $x_i = i,\ i = 0, 1, 2, 3$ with a linear spline $s \in S_{1,\Delta}$.

5. Let $f(x) = x^5 + x^4$, $x \in [-2, 2]$. Consider the grid $\{-2, -1, 0, 1, 2\}$ as a partition of $[-2, 2]$ and determine the natural cubic spline $s \in S_{3,\Delta}$ that interpolates f at the nodal points $-2, -1, 0, 1, 2$.

   Hint: You can use MATLAB *to solve the linear system for the moments of* s *and plot* f *and* s.

6. Create a MATLAB-Function that implements the linear spline interpolation for a given function $f : [a, b] \to \mathbb{R}$ and any grid Δ on the interval $[a, b]$. The algorithm should have the following structure:

   INPUT: Δ, i.e. the number of grid points.
   MAIN BODY: Algorithm based on the polynomial of exercise 4.
   OUTPUT: Graph of f and $s \in S_{1,\Delta}$.

   Consider the function

$$f(x) = \frac{1}{1 + 25x^2}, \quad x \in [-1, 1],$$

   and $n = 4, 8, 12$ equidistant grid points.

7. An efficient implementation in MATLAB to calculate the Fourier coefficients a_k of the DFT

$$x_n = \sum_{k=0}^{N-1} a_k e^{ik\frac{2\pi}{N}n}, \quad n = 0, ..., N-1$$

   is the function `fft`. To illustrate some of the properties of the DFT, consider different rectangular pulses of the form

$$x_n = \begin{cases} 1, & n \in [\frac{N-1}{2} - a, \frac{N-1}{2} + a] \\ 0, & \text{otherwise} \end{cases},$$

   for a given positive integer a and N odd number. Additionally, consider the pulses,

$$x_{bn} = \begin{cases} 1, & n \in [\frac{N-1}{2} - a/b, \frac{N-1}{2} + a/b] \\ 0, & \text{otherwise} \end{cases},$$

   and

$$x_n^{(b)} = \begin{cases} 1, & n \in [\frac{N-1}{2} - ab : b : \frac{N-1}{2} + ab] \\ 0, & \text{otherwise} \end{cases},$$

   related to decimation and time expansion, respectively. Using `fft` for $N = 31$, $a = 2$ and $b = 1, 2, 3$ present the Fourier transforms (using the graphs of a_k) for the different values of b. Additionally, prove the result for the case of x_{3n}.