Exercise Sheet 3

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function satisfying $f(a) \cdot f(b) < 0$. We define the Bisection method, to approximate the solution of the equation $f(x) = 0$, as:

ALGORITHM (BISECTION METHOD)

`For` $k = 1, 2, \ldots$

- `Set`
$$x_k = a + \frac{b-a}{2}$$
- `If` $f(x_k) = 0$
`then` x_k `is the solution.`
`end if.`
- `If` $f(a) \cdot f(x_k) > 0$
$$a = x_k$$
`else`
$$b = x_k$$
`end if.`

`end for.`

(a) Given that the sequence $\{x_k\}_{k=1}^{\infty}$ converges to the solution $x^* \in (a, b)$ of $f(x) = 0$, show that

$$|x_k - x^*| \leq \frac{b-a}{2^k}, \quad k = 1, 2, ...$$

(b) Determine the number of iteration steps required for approximating x^* with tolerance $10^{-\alpha}$.

2. Consider the function $f(x) = x^3 - 2x - 5$. Approximate the solution of the equation $f(x) = 0$, using the first three steps of the:

   (a) Bisection method at the Interval $[2, 3]$,

   (b) Secant method with $x_0 = 3$ and $x_1 = 3.5$,

   (c) Newton method with $x_0 = 3$.

3. To approximate the solutions of the equation $x^2 - x - 2 = 0$ we can rewrite it in two different forms:

   (a) $x = x^2 - 2 := \phi_1(x)$,

   (b) $(x^2 - x - 2)/x = 0 \quad \Rightarrow \quad x = 1 + 2/x := \phi_2(x), \quad x \neq 0$

   and we consider the fixed-point method for $j = 1, 2$,

$$x_{n+1} = \phi_j(x_n), \quad n = 0, 1, ...$$

   Setting $x_0 = -3$, perform the first four steps for both iteration functions ϕ_j and analyse the convergence of the method.

4. Consider the following system of equations

$$f(x, y) := \begin{pmatrix} xy \\ xy^2 + x - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   Perform the first two steps of the Newton method with initial vector $(x_0, y_0) = (1/2, 1)$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{1 + x^2}$. Show that the Newton method for the equation $f'(x) = 0$ converges to the exact solution $x^* = 0$ if the initial guess satisfies $|x^{(0)}| < 1$.

6. Assume $f : \mathbb{R} \to \mathbb{R}$ to be a three times continuously differentiable function and x^* one of its zeros. Consider the following iteration method,

$$x_{n+1} = x_n - \frac{f(x_n)}{g(x_n)}, \quad \text{where} \quad g(x) = \frac{f(x + f(x)) - f(x)}{f(x)}$$

   to approximate x^*. Implement in MATLAB the above method for solving the equation $e^{-x} - \sin(x) = 0$.

7. Consider the system of equations

$$f(x, y, z) := \begin{pmatrix} xy - z^2 - 1 \\ xyz - x^2 + y^2 + 2 \\ e^x - e^y + z - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Implement in MATLAB the Broyden's method to approximate the solution of the above system with initial guess $\mathbf{x}_0 = (x_0, y_0, z_0) = (1, 1, 1)^T$ and the exact Jacobian $B_0 = J_f(\mathbf{x}_0)$.